Exercise 2.3.3

(Tumor Growth) The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

a) Interpret a and b biologically.

b) Sketch the vector field and graph $N(t)$ for various initial values.

The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

Solution

The Gompertz law governs the population growth of cancer cells.

$$\dot{N} = -aN\ln(bN)$$

The larger a is, the faster the population grows and settles to equilibrium. The smaller b is, the higher the carrying capacity and the faster the population grows and settles to equilibrium. The fixed points occur where $\dot{N} = 0$.

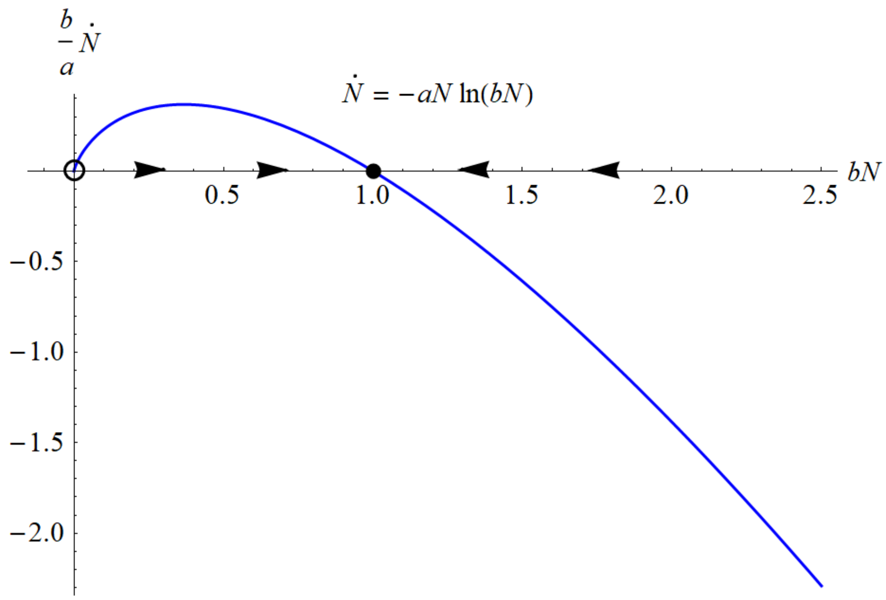
$$-aN^*\ln(bN^*) = 0$$

$$-aN^* = 0 \quad \text{or} \quad \ln(bN^*) = 0$$

$$N^* = 0 \quad \text{or} \quad bN^* = 1$$

$$N^* = 0 \quad \text{or} \quad N^* = \frac{1}{b}$$

Plot $(b/a)\dot{N}$ versus bN in order to determine the stability of these fixed points.



When the function is negative the flow is to the left, and when the function is positive the flow is to the right. Therefore, $N^* = 0$ is locally unstable, and $N^* = 1/b$ is locally stable.

Below is a qualitative sketch of N versus t for various initial conditions.